MATH 302 Applied Project: Approximation Theory and Related Applications

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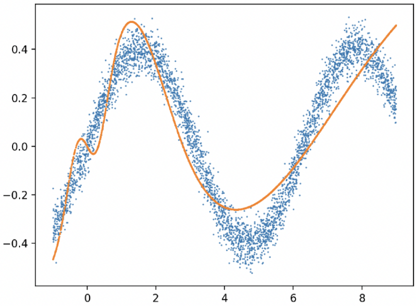
Outline

1. Background of approximation theory
   - Discrete least squares approximation
   - Continuous least squares approximation
   - Other techniques (least squares using Legendre/Chebyshev polynomials)
2. Research questions
   - How to find the "line" of best fit in practical problems?
   - Can we identify the trends in stocks and make prediction about the trends using least squares approximation?
3. Results
   - Market price of gold vs Stock price of a gold mining company
   - Weather forecast
   - Identify the trends in stocks (Original result)
4. Summary

Background of approximation theory

- Approximation theory is the branch of mathematics which studies the process of approximating general functions by simple functions such as polynomials.
- It therefore plays a central role in the analysis of numerical methods.



https://gaoxiangluo.github.io/2020/09/27/Visual-and-Rigorous-Proof-of-Universal-Approximation-Theorem-UAT/

Discrete least squares approximation

Discrete least squares approximation

- Linear least squares
- Polynomial least squares
- Non-polynomials least squares

Discrete least squares approximation (Linear)

Linear least squares

- Data: $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$
- Linear approximation: $y = f(x) = ax + b$
- Find a, b that minimize the error $E = \sum_{i=1}^{m}(y_i - ax_i - b)^2$
- For a minimum to occur, we must have $\frac{\partial E}{\partial a} = 0$ and $\frac{\partial E}{\partial b} = 0$
- We have:

$$\frac{\partial E}{\partial a} = \sum_{i=1}^{m} \frac{\partial E}{\partial a}(y_i - ax_i - b)^2 = \sum_{i=1}^{m}(-2x_i)(y_i - ax_i - b) = 0$$
$$\frac{\partial E}{\partial b} = \sum_{i=1}^{m} \frac{\partial E}{\partial b}(y_i - ax_i - b)^2 = \sum_{i=1}^{m}(-2)(y_i - ax_i - b) = 0$$

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$$\frac{\partial E}{\partial b} = \sum_{i=1}^{m} \frac{\partial E}{\partial b}(y_i - ax_i - b)^2 = \sum_{i=1}^{m}(-2)(y_i - ax_i - b) = 0$$

P.S. The error term E has the same idea as the **residual vector** we learned on Wednesday.

Definition 7.23 Suppose $\tilde{\mathbf{x}} \in \mathbb{R}^n$ is an approximation to the solution of the linear system defined by $A\mathbf{x} = \mathbf{b}$. The **residual vector** for $\tilde{\mathbf{x}}$ with respect to this system is $\mathbf{r} = \mathbf{b} - A\tilde{\mathbf{x}}$. ■

Textbook, p. 469

Discrete least squares approximation (Linear)

Linear least squares (continued)

- Using algebra, these equations can be simplified as

$$b\sum_{i=1}^{m} x_i + a\sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i y_i$$
$$bm + a\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i,$$

- The solution to this system of equations is

$$a = \frac{m\sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i}{m\left(\sum_{i=1}^{m} x_i^2\right) - \left(\sum_{i=1}^{m} x_i\right)^2}, b = \frac{\sum_{i=1}^{m} x_i^2 \sum_{i=1}^{m} y_i - \sum_{i=1}^{m} x_i y_i \sum_{i=1}^{m} x_i}{m\left(\sum_{i=1}^{m} x_i^2\right) - \left(\sum_{i=1}^{m} x_i\right)^2}.$$

- Now we get a, b that minimize the error $E = \sum_{i=1}^{m}(y_i - ax_i - b)^2$

Discrete least squares approximation (Polynomial)

Polynomial least squares

- Data: $(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$
- Polynomial approximation: $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ where m will be usually much larger than n and $a_n \neq 0$
- Similarly, we want to minimize

$$E = \sum_{i=1}^{m} (y_i - P_n(x_i))^2 = \sum_{i=1}^{m} \left(y_i - \sum_{j=0}^{n} a_j x_i^j \right)^2$$

with respect to the parameters $a_n, a_{n-1}, \dots, a_0$

- For a minimum to occur, the necessary conditions are

$$\frac{\partial E}{\partial a_k} = 0 \Rightarrow -\sum_{i=1}^{m} y_i x_i^k + \sum_{j=0}^{n} a_j \left(\sum_{i=1}^{m} x_i^{k+j} \right) = 0$$

for $k = 0, 1, \dots, n$

Discrete least squares approximation (Polynomial)

Polynomial least squares (continued)

- Using algebra, we get a system of $(n+1)$ equations and $(n+1)$ unknowns

$$\sum_{j=0}^{n} a_j \left(\sum_{i=1}^{m} x_i^{k+j} \right) = \sum_{i=1}^{m} y_i x_i^k$$

  for $k = 0, 1, \ldots, n$

- We can write this system as a matrix equation

$$\mathbf{A}a = \mathbf{b}$$

  where a is the unknown vector we are trying to find, and b is the constant vector

Discrete least squares approximation (Polynomial)

Polynomial least squares (continued)

- where a is the unknown vector we are trying to find, and b is the constant vector

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, b = \begin{bmatrix} \sum_{i=1}^m y_i \\ \sum_{i=1}^m y_i x_i \\ \vdots \\ \sum_{i=1}^m y_i x_i^n \end{bmatrix}$$

  and A is an $(n+1)$ by $(n+1)$ symmetric matrix with (kj)th entry A_{kj}, $k = 1, 2, \ldots, n+1$, $j = 1, 2, \ldots, n+1$ given by

$$A_{kj} = \sum_{i=1}^{m} x_i^{k+j-2}$$

- The matrix equation $Aa = b$ has a unique solution if the x_i are distinct, and $n \leq m - 1$

Discrete least squares approximation (Non-polynomials)

Non-polynomials least squares

- The method of least squares is not only for polynomials. For example, suppose we want to find the function

$$f(t) = a + bt + c\sin(2\pi t/365) + d\cos(2\pi t/365)$$

  that has the best fit to some data $(t_1, T_1), \ldots, (t_m, T_m)$ in the least-squares sense. This function is used in modeling weather temperature data, where t denotes time, and T denotes the temperature.
- Similarly, we want to minimize the least squares error term

$$E = \sum_{i=1}^{m}(f(t_i) - T_i)^2 = \sum_{i=1}^{m}\left(a + bt_i + c\sin(\frac{2\pi t_i}{365}) + d\cos(\frac{2\pi t_i}{365}) - T_i\right)^2$$

Discrete least squares approximation (Non-polynomials)

Non-polynomials least squares (continued)

- We set its partial derivatives with respect to the unknowns *a*, *b*, *c*, *d* to zero to obtain the following equations:

$$\frac{\partial E}{\partial a} = 0 \Rightarrow \sum_{i=1}^{m} 2\left(a + bt_i + c\sin\left(\frac{2\pi t_i}{365}\right) + d\cos\left(\frac{2\pi t_i}{365}\right) - T_i\right) = 0$$
$$\Rightarrow \sum_{i=1}^{m} \left(a + bt_i + c\sin\left(\frac{2\pi t_i}{365}\right) + d\cos\left(\frac{2\pi t_i}{365}\right) - T_i\right) = 0,$$

$$\frac{\partial E}{\partial b} = 0 \Rightarrow \sum_{i=1}^{m} (2t_i)\left(a + bt_i + c\sin\left(\frac{2\pi t_i}{365}\right) + d\cos\left(\frac{2\pi t_i}{365}\right) - T_i\right) = 0$$
$$\Rightarrow \sum_{i=1}^{m} t_i\left(a + bt_i + c\sin\left(\frac{2\pi t_i}{365}\right) + d\cos\left(\frac{2\pi t_i}{365}\right) - T_i\right) = 0,$$

$$\frac{\partial E}{\partial c} = 0 \Rightarrow \sum_{i=1}^{m} \left(2\sin\left(\frac{2\pi t_i}{365}\right)\right)\left(a + bt_i + c\sin\left(\frac{2\pi t_i}{365}\right) + d\cos\left(\frac{2\pi t_i}{365}\right) - T_i\right) = 0$$
$$\Rightarrow \sum_{i=1}^{m} \sin\left(\frac{2\pi t_i}{365}\right)\left(a + bt_i + c\sin\left(\frac{2\pi t_i}{365}\right) + d\cos\left(\frac{2\pi t_i}{365}\right) - T_i\right) = 0,$$

Discrete least squares approximation (Non-polynomials)

$$\frac{\partial E}{\partial d} = 0 \Rightarrow \sum_{i=1}^{m} \left(2\cos\left(\frac{2\pi t_i}{365}\right)\right)\left(a + bt_i + c\sin\left(\frac{2\pi t_i}{365}\right) + d\cos\left(\frac{2\pi t_i}{365}\right) - T_i\right) = 0$$
$$\Rightarrow \sum_{i=1}^{m} \cos\left(\frac{2\pi t_i}{365}\right)\left(a + bt_i + c\sin\left(\frac{2\pi t_i}{365}\right) + d\cos\left(\frac{2\pi t_i}{365}\right) - T_i\right) = 0.$$

Non-polynomials least squares (continued)

- Rearranging terms in the above equations, we get a system of four equations and four unknowns:

$$am + b\sum_{i=1}^{m} t_i + c\sum_{i=1}^{m} \sin\left(\frac{2\pi t_i}{365}\right) + d\sum_{i=1}^{m} \cos\left(\frac{2\pi t_i}{365}\right) = \sum_{i=1}^{m} T_i$$
$$a\sum_{i=1}^{m} t_i + b\sum_{i=1}^{m} t_i^2 + c\sum_{i=1}^{m} t_i\sin\left(\frac{2\pi t_i}{365}\right) + d\sum_{i=1}^{m} t_i\cos\left(\frac{2\pi t_i}{365}\right) = \sum_{i=1}^{m} T_i t_i$$
$$a\sum_{i=1}^{m} \sin\left(\frac{2\pi t_i}{365}\right) + b\sum_{i=1}^{m} t_i\sin\left(\frac{2\pi t_i}{365}\right) + c\sum_{i=1}^{m} \sin^2\left(\frac{2\pi t_i}{365}\right) + d\sum_{i=1}^{m} \sin\left(\frac{2\pi t_i}{365}\right)\cos\left(\frac{2\pi t_i}{365}\right)$$
$$= \sum_{i=1}^{m} T_i\sin\left(\frac{2\pi t_i}{365}\right)$$
$$a\sum_{i=1}^{m} \cos\left(\frac{2\pi t_i}{365}\right) + b\sum_{i=1}^{m} t_i\cos\left(\frac{2\pi t_i}{365}\right) + c\sum_{i=1}^{m} \sin\left(\frac{2\pi t_i}{365}\right)\cos\left(\frac{2\pi t_i}{365}\right) + d\sum_{i=1}^{m} \cos^2\left(\frac{2\pi t_i}{365}\right)$$
$$= \sum_{i=1}^{m} T_i\cos\left(\frac{2\pi t_i}{365}\right)$$

Discrete least squares approximation (Non-polynomials)

Non-polynomials least squares (continued)

- Using a short-hand notation where we suppress the argument $\left(\frac{2\pi t_i}{365}\right)$ in the trigonometric functions, and the summation indices, we write the above equations as a matrix equation:

$$\underbrace{\begin{bmatrix} m & \sum t_i & \sum \sin(\cdot) & \sum \cos(\cdot) \\ \sum t_i & \sum t_i^2 & \sum t_i \sin(\cdot) & \sum t_i \cos(\cdot) \\ \sum \sin(\cdot) & \sum t_i \sin(\cdot) & \sum \sin^2(\cdot) & \sum \sin(\cdot)\cos(\cdot) \\ \sum \cos(\cdot) & \sum t_i \cos(\cdot) & \sum \sin(\cdot)\cos(\cdot) & \sum \cos^2(\cdot) \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \underbrace{\begin{bmatrix} \sum T_i \\ \sum T_i t_i \\ \sum T_i \sin(\cdot) \\ \sum T_i \cos(\cdot) \end{bmatrix}}_{\mathbf{r}}$$

- Solve the equation $\mathbf{A}x = \mathbf{r}$ to get the unknowns a, b, c, d

Continuous least squares approximation

Continuous least squares approximation

- In discrete least squares, our starting point was a set of data points. Here we will start with a continuous function f on $[a, b]$ and answer the following question: how can we find the "best" polynomial $P_n(x) = \sum_{j=0}^{n} a_j x^j$ of degree at most n, that approximates f on $[a, b]$?
- As before, "best" polynomial will mean the polynomial that minimizes the least squares error:

$$E = \int_a^b \left(f(x) - \sum_{j=0}^{n} a_j x^j \right)^2 dx$$

- Compare this expression with that of the discrete least squares:

$$E = \sum_{i=1}^{m} \left(y_i - \sum_{j=0}^{n} a_j x_i^j \right)^2$$

Continuous least squares approximation

Continuous least squares approximation (continued)

- To minimize E we set $\frac{\partial E}{\partial a_k} = 0$, for $k = 0, 1, \dots, n$, and observe

$$\frac{\partial E}{\partial a_k} = \frac{\partial}{\partial a_k}\left(\int_a^b f^2(x)dx - 2\int_a^b f(x)\left(\sum_{j=0}^n a_j x^j\right)dx + \int_a^b \left(\sum_{j=0}^n a_j x^j\right)^2 dx\right)$$
$$= -2\int_a^b f(x)x^k dx + 2\sum_{j=0} a_j \int_a^b x^{j+k} dx = 0,$$

  which gives the $(n+1)$ equations for the continuous least squares problem:

$$\sum_{j=0}^n a_j \int_a^b x^{j+k} dx = \int_a^b f(x)x^k dx$$

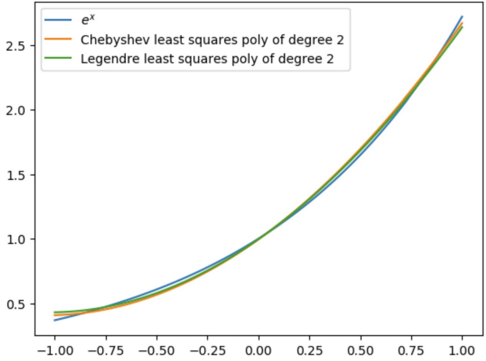
  for $k = 0, 1, \dots, n$

- Note that the only unknowns in these equations are the a_j's, hence this is a linear system of equations.

Other techniques

Other techniques

- Least squares using Legendre polynomials
- Least squares using Chebyshev polynomials



Compare the quadratic approximations obtained by Legendre and Chebyshev polynomials. We can see visually that Chebyshev does a better approximation at the end points of the interval (*First Semester Numerical Analysis with Julia*, p. 206).

Research questions

Research questions

1. How to find the “line” of best fit in practical problems?
2. Can we identify the trends in stocks and make prediction about the trends using least squares approximation?

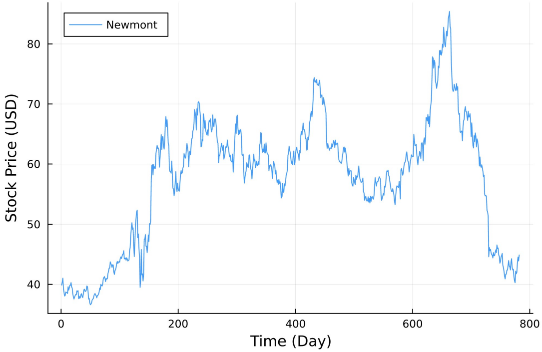
https://ohiostate.pressbooks.pub/choosingsources/chapter/purpose-of-research-questions/

Results (Part I)

A practical problem

- There is an investment considering whether to invest in a gold mining company or not. We want to know how sensitive the company's stock price is to changes in the market price of gold.
- To study this, we could use the least squares method to trace the relationship between those two variables over time onto a scatter plot. This analysis could help us predict the degree to which the stock's price would likely rise or fall for any given change in the price of gold.
- In this project, we trace the stock price of Newmont, which is one of the world's largest gold mining corporations, and the market price of gold from 2019/8/30 to 2022/10/6.

Results (Part I)



Plotted by Julia

https://finance.yahoo.com/news/newmont-delays-investment-decision-peru-135023690.html

Results (Part I)

What we understand from the analysis

- The stock price of Newmont rises as the market price of gold goes up.
- The relationship is not simply linear.
- Investors should pay attention to the change of gold price when investing gold mining companies.

Results (Part II)

Apply non-polynomials least squares to weather forecast

$$f(t) = a + bt + c\sin(\frac{2\pi t}{365}) + d\cos(\frac{2\pi t}{365})$$

- The period T is 365 days.
- The term bt implies "global warming" if $b > 0$.

Results (Part III)

How to apply non-polynomials least squares to the stock market

$$f(t) = a + bt + c\sin\left(\frac{2\pi t}{T}\right) + d\cos\left(\frac{2\pi t}{T}\right)$$

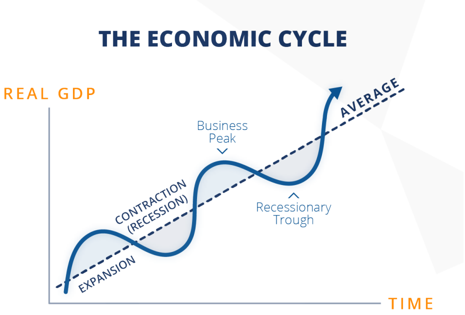
Questions we need to answer

- What is the period T in this case? Is it still 365 days?
- What does the term bt represent in the stock market?

Results (Part III)

To answer the above questions, we need to know

- The economic cycle



https://www.fe.training/free-resources/asset-management/stages-of-the-economic-cycle/

- The average length of recessions in the U.S. since World War II has been just around 11 months. U.S. expansions have typically lasted longer than U.S. recessions.

https://www.kiplinger.com/slideshow/investing/t038-s001-recessions-10-facts-you-must-know/index.html

Results (Part III)

To answer the above questions, we need to know

- Cyclical stocks:

  Cyclical businesses perform well during economic expansions but typically experience significantly declining sales and profits during recessions.
- Cyclical stocks are generally concentrated in specific industries such as airlines, real estate, and construction.

https://finbold.com/guide/cyclical-stocks-definition/

Results (Part III)

Now we are able to apply non-polynomials least squares

- We trace the stock price of Lennar, which is the leading homebuilder in the U.S., from 2012/7/2 to 2018/9/27.
- The period T is approximately 22 months.
- The term bt shows that the real estate industry develops as the U.S. economy grows (the stock trend is going up).

Results (Part III)

Now we are able to apply non-polynomials least squares

- We trace the stock price of Lennar, which is the leading homebuilder in the U.S., from 2012/7/2 to 2018/9/27.
- The period T is approximately 22 months.
- The term bt shows that the real estate industry develops as the U.S. economy grows (the stock trend is going up).

Results (Part III)

Some limitations on our approach

- We can only investigate the cyclical stocks.
- The trend in the stock market (e.g., S & P 500) is supposed to be going up or going down (there exists a general trend).
- "Black swan events" may influence the approximation significantly.

P.S. The limitations are not always a bad thing. Knowing the limitations of a method actually enables us to apply it properly and effectively.

Summary

1. Approximation theory
   - Discrete least squares approximation
   - Continuous least squares approximation
   - Other techniques (least squares using Legendre/Chebyshev polynomials)
2. Research questions
   - How to find the "line" of best fit in practical problems?
   - Can we identify the trends in stocks and make prediction about the trends using least squares approximation?
3. Results
   - The "line" of best fit of gold price vs stock price
   - Weather forecast using non-polynomials least squares
   - Identify the trends in stocks using non-polynomials least squares
4. Summary

References

References:

- Numerical Analysis 10E
- First Semester Numerical Analysis with Julia
- Cyclical stocks: https://finbold.com/guide/cyclical-stocks-definition/
- The economic cycle: https://www.kiplinger.com/slideshow/investing/t038-s001-recessions-10-facts-you-must-know/index.html

The end

Thanks for your listening!