Applied Mathematics and Computational Sciences

MATH 303: ODE and Dynamical Systems

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Modeling the COVID-19 Epidemic with ODE

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Many persons who have not studied mathematics confuse it with arithmetic and consider it a dry and arid science. Actually, however, this science requires great fantasy. – Sophia Kovalevsky

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1 Introduction

The subject of infectious diseases has been one of the richest areas of application of mathematics in biology (Brauer et al. 2019). Infectious diseases obviously impose great burdens of morbidity and mortality on humanity, as well as on non-human populations of importance to us. Communicable diseases have played a significant role in shaping human history. The Black Death spread starting in 1346 and has been estimated to have caused the death of as much as one-third of the population of Europe between 1346 and 1350. Therefore, it is no wonder that the ability to use formal models to reduce those burdens has attracted the attention of many mathematicians.

First discovered in Southern Africa in November 2021, the Omicron variant of SARS-CoV-2 has spread swiftly across the world and replaced the Delta variant to become the dominant strain globally (Cai et al. 2022). Having adopted a dynamic zero-COVID strategy to respond to SARS-CoV-2 variants with higher transmissibility since August 2021, China is now easing most of its Covid restrictions, in significant step towards reopening. Therefore, it is essential now to improve and perfect the strategies for minimizing disruption to the healthcare system in the case of a nationwide epidemic.

In the face of the COVID-19 epidemic, a wide variety of SIR models of the progression of this epidemic are being used by public health experts to generate scenarios that are being used to guide decisions to recommend and impose proper measures to contain the epidemic worldwide.

In this project, we are going to discuss thoroughly the SIR model and some variants of it in epidemiology. We also combine several variants together to get a relatively new model, which takes more parameters into account. Note that generally speaking introducing more parameters to an epidemiological model tends to provide us with more accuracy and flexibility.

The main questions we would like to answer are as follows:

(1) How do ODE models help us understand the epidemic?
(2) How will social distancing and vaccination help contain the epidemic?

There are basically three results we would like to present. The first one is that we analyse the properties and dynamics of the SEIR model. The second one is that we implement the SEIR model with social distancing and vaccination using Python. The last one is that we discuss several drawbacks of the ODE models we use and provide some strategies for minimizing disruption to the healthcare system base on the ODE models.

2 Preliminaries

2.1 The SIR model

The SIR model for an epidemic addresses the spread of diseases that are only contracted by contact with an infected individual; its victims, once recovered, are immune to further infection and are themselves noninfectious. So the members of a population of size N fall into three classes as we have discussed in class before:

- $S(t)$ = the number of susceptible individuals—that is, those who have not been infected; $s := S/N$ is the fraction of susceptibles.
- $I(t)$ = the number of individuals who are currently infected, comprising a fraction $i := I/N$ of the population.
- $R(t)$ = the number of individuals who have recovered from infection, comprising the fraction $r := R/N$.

The classic SIR epidemic model assumes that on the average, an infectious individual encounters a people per unit time (usually per week). Thus, a total of aI people per week are contacted by infectees, but only a fraction $s = S/N$ of them are susceptible (Nagle et al. 2018). Based on this, we will have the following equations which is the so-called SIR model in epidemiology:

$$\frac{dS}{dt} = -\frac{\beta IS}{N}$$
$$\frac{dI}{dt} = \frac{\beta IS}{N} - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

Note that we have $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$. The parameter β is crucial in disease control. Crowded conditions, or high β, make it difficult to combat the spread of infection. Ideally, we would quarantine the infectees (low β) to fight the epidemic.

2.2 Some variants of the SIR model

The second model we would like to mention is the SEIR model. We will do here is introduce another state called 'exposed' to the original SIR model, which actually makes sense because for a susceptible individual to transfer to the 'infected' state he or she will be first exposed to some infected persons and then get infected. Therefore, we are able to get the following ODE system which is the so-called SEIR model:

$$\frac{dS}{dt} = -\frac{\beta IS}{N}$$
$$\frac{dE}{dt} = \frac{\beta IS}{N} - \sigma E$$
$$\frac{dI}{dt} = \sigma E - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

Note that we have $\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$. To apply the above model practically, we need to know the following approximations:

- The infection rate $\sigma \approx \frac{1}{\text{incubation period}}$. The incubation periods of COVID-19 caused by the Alpha, Beta, Delta, and Omicron variants were 5.00, 4.50, 4.41, and 3.42 days, respectively (Wu et al. 2022).
- The recovery rate $\gamma \approx \frac{1}{\text{duration infection}}$. As far as how long the Omicron symptoms last, research shows that people have acute symptoms for about six to seven days—about two days shorter than Delta's eight to nine days of acute illness.
- The basic reproduction number $R_0 \approx \frac{\beta}{\gamma}$ where β is the effective contact rate. The Omicron variant has an average basic reproduction number of 9.5 and a range from 5.5 to 24.

The next two models we want to introduce are based on the SEIR model. It's reasonable to introduce more parameters to the SEIR model and the first parameter we are going to consider is social distancing. The SEIR model with social distancing can be expressed by the following ODE system:

$$\frac{dS}{dt} = -(1-\mu)\frac{\beta IS}{N}$$
$$\frac{dE}{dt} = (1-\mu)\frac{\beta IS}{N} - \sigma E$$

$$\frac{dI}{dt} = \sigma E - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

Note that we still have $\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$. The new parameter μ is essential in the above model. μ describes the effectiveness on any public health interventions to control transmission of the disease. We have $\mu \in [0, 1]$. $\mu = 0$ corresponds to no effective public health interventions, $\mu = 1$ implies total elimination of disease transmission, which means complete isolation.

If we introduce another parameter that represents vaccination to the SEIR model, together with social distancing, we will get the following ODE system:

$$\frac{dS}{dt} = -(1-\mu)\frac{\beta IS}{N} - C$$

$$\frac{dE}{dt} = (1-\mu)\frac{\beta IS}{N} - \sigma E$$

$$\frac{dI}{dt} = \sigma E - \gamma I$$

$$\frac{dR}{dt} = \gamma I - C$$

Note that we have $\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$ as before. The new parameter C represents the number of susceptible individuals who get vaccinated in one day. $C(t)$ can be a function of time t, but for simplicity we assume $C(t)$ is a positive constant in the next section. One important assumption in the above model is that once the susceptible individuals get vaccinated, they are assumed to have acquired immunity. Therefore, those who get vaccinated will transfer to the 'removed' state from the 'susceptible' state. We would like to point out that the SEIR model with both social distancing and vaccination is a simple and relatively new ODE model, which can help us have a better understanding of how to contain the epidemic.

What should be emphasized is that all of the above ODE models for the epidemic are based on ideal assumptions. The real situation is much more complicated.

3 Results

In this section, we will present the results in three parts. The first part is to discuss the properties and dynamics of the SEIR model. The second part is to implement the SEIR model with social distancing and vaccination using Python. The last part is to discuss the drawbacks of the ODE models and provide some strategies for containing the epidemic effectively and at relatively low cost.

3.1 Properties and dynamics of SEIR model

We consider the development of the epidemic as three different periods – start, reaching the maximum, and the end.

- Case 1: $\frac{S}{N} \approx 1$ (start).

Since we are more interested in the trends in the number of infected individuals, we are going to investigate the following two ODEs:

$$\frac{dE}{dt} = \beta I - \sigma E$$

$$\frac{dI}{dt} = -\gamma I + \sigma E$$

.

Note that we have a linear system with constant coefficients. Hence we are able to apply the lemma we have learned before.

Lemma 1. Suppose that $\boldsymbol{A}$ has a pair of distinct real eigenvalues, λ_1 and λ_2, with associated eigenvectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. Then the general solution of the linear system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is given by

$$\boldsymbol{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{v_1} + c_2 e^{\lambda_2 t}\boldsymbol{v_2}.$$

The linear system we have is equivalent to

$$\frac{d}{dt}\begin{bmatrix} E \\ I \end{bmatrix} = \begin{bmatrix} -\sigma & \beta \\ \sigma & -\gamma \end{bmatrix}\begin{bmatrix} E \\ I \end{bmatrix}.$$

Now we are going to investigate the characteristic polynomial

$$(-\sigma - \lambda)(-\gamma - \lambda) - \sigma\beta = 0,$$

which implies

$$\lambda^2 + (\sigma + \gamma)\lambda - \sigma(\beta - \gamma) = 0.$$

Therefore, we get the discriminant

$$\Delta = (\sigma + \gamma)^2 + 4\sigma(\beta - \gamma) = (\sigma - \gamma)^2 + 4\sigma\beta > 0.$$

Then we have two real eigenvalues

$$\lambda_+\lambda_- = -\sigma(\beta - \gamma) < 0.$$

As we mentioned previously, we have the approximation $\beta \approx R_0\gamma$ and for an epidemic we always have the basic reproduction number $R_0 > 1$. Therefore, we get $\beta > \gamma$ and this is the reason why the discriminant is larger than 0.

Therefore, what we can conclude now is that according to Lemma 1 we will have exponential growth in E and I due to the positive λ_+.

- Case 2: $\frac{S}{N} \approx 0$ (end).

Correspondingly, we have the following ODE system, which is also linear:

$$\frac{dE}{dt} = -\sigma E$$
$$\frac{dI}{dt} = -\gamma I + \sigma E$$

Similarly, we write the linear system in the following way:

$$\frac{d}{dt}\begin{bmatrix} E \\ I \end{bmatrix} = \begin{bmatrix} -\sigma & 0 \\ \sigma & -\gamma \end{bmatrix}\begin{bmatrix} E \\ I \end{bmatrix}.$$

It is easy to get $\lambda_1 = -\sigma$, $\lambda_2 = -\gamma$. Therefore, according to Lemma 1 we will have exponential decay in E and I due to negative eigenvalues.

- Case 3: $\frac{dE}{dt} \approx 0$, $\frac{dI}{dt} \approx 0$ (I and E stop growing; herd immunity is reached).

Another thing we care about in the SEIR model is when the number of infected individuals per day will stop growing. Therefore, we have $\frac{dE}{dt} \approx 0$ and $\frac{dI}{dt} \approx 0$.

Correspondingly, we have the following two equations:

$$\gamma I \approx \sigma E$$
$$\beta S \frac{I}{N} \approx \sigma E$$

As a result, we have

$$\gamma I \approx \beta S \frac{I}{N},$$

which implies the relation between S and N can be expressed by

$$\frac{S}{N} \approx \frac{\gamma}{\beta}.$$

Now we know when the number of infected individuals per day reaches its maximum, we have that $\frac{S}{N}$ is approximately equal to $\frac{\gamma}{\beta}$, which is a constant.

Given some initial conditions, we plot the trends in the number of exposed individuals and infected individuals using Python.

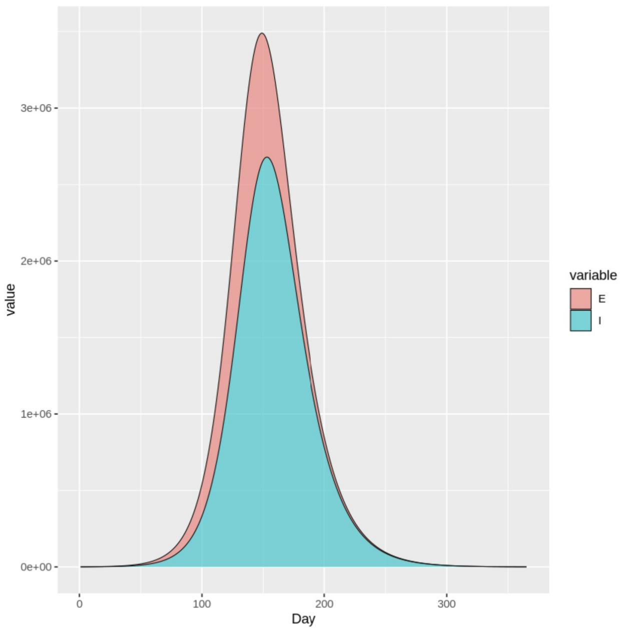

Figure 1: Trends in the number of exposed and infected individuals.

Source: Plotted by Python

To sum up, according to the above analysis, the number of exposed individuals and infected individuals will grow exponentially at start, reach the maximum when herd immunity is achieved, and finally decay exponentially, which agrees with what we see in Figure 1.

Based on what we have learned so far, we can also conclude that the only equilibrium $(0, 0)$ in Case 1 is a saddle since we have $\lambda_- < 0 < \lambda_+$ and in Case 2 the equilibrium $(0, 0)$ is a stable node since we have two negative eigenvalues.

For the Hopf bifurcation of the variants of the SIR model, one can refer to the paper by Ajbar et al. (2021). It turns out that when the basic reproduction number R_0 is less than unity the model can exhibit a number of nonlinear phenomena including saddle-node, backward, and Hopf bifurcations.

3.2 Implementation of SEIR model with social distancing and vaccination

Now we will implement the SEIR model with social distancing using Python. Note that larger μ in the model implies stricter rules of social distancing. For instance, $\mu = 0.1$ may represent wearing masks and $\mu = 0.2$ may represent wearing masks and having hybrid classes. We will have the following plots:

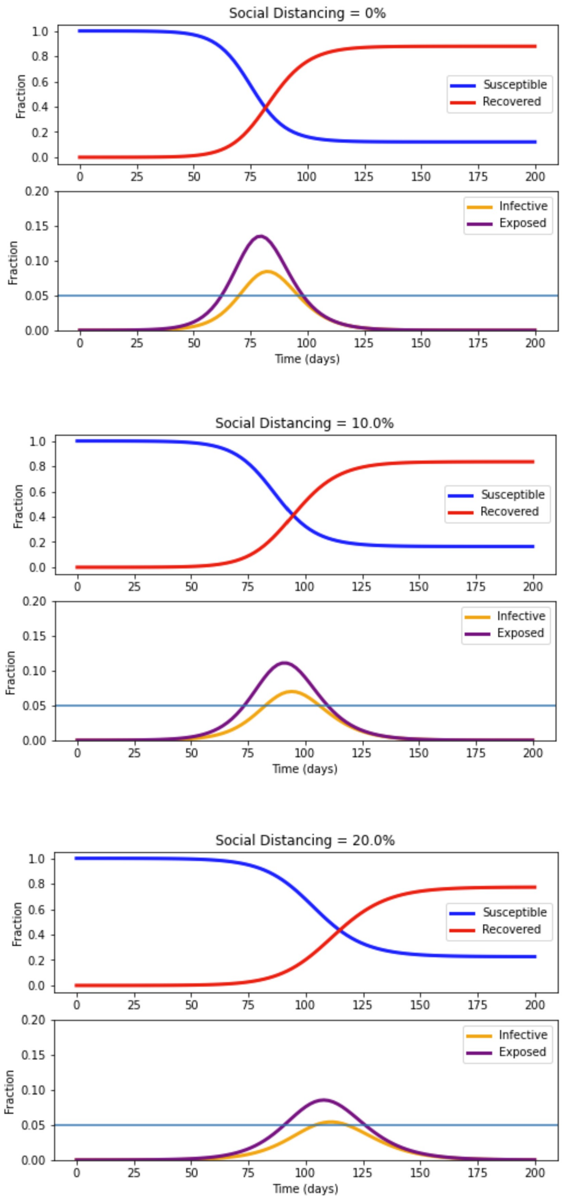

Figure 2: Social distancing parameter $\mu = 0, 0.1, 0.2$.

Source: Plotted by Python

As we can see in Figure 2, larger μ will result in smaller value of the number of infected individuals per day. When $\mu = 0$ suggesting there is no social distancing, the maximum of $\frac{I}{N}$ is close to 0.1. When $\mu = 0.2$, the maximum of $\frac{I}{N}$ is about 0.05, which is relative small.

Now we are going to implement the SEIR model with both social distancing and vaccination using Python. Assume that $C = 0.1\% N$, which means 0.1% of the population will get vaccinated everyday. We compare the situation where we only have the parameter $\mu = 0.1$ with the situation where the parameter $\mu = 0.1$ and $C = 0.1\% N$. What we have plotted is as follows:

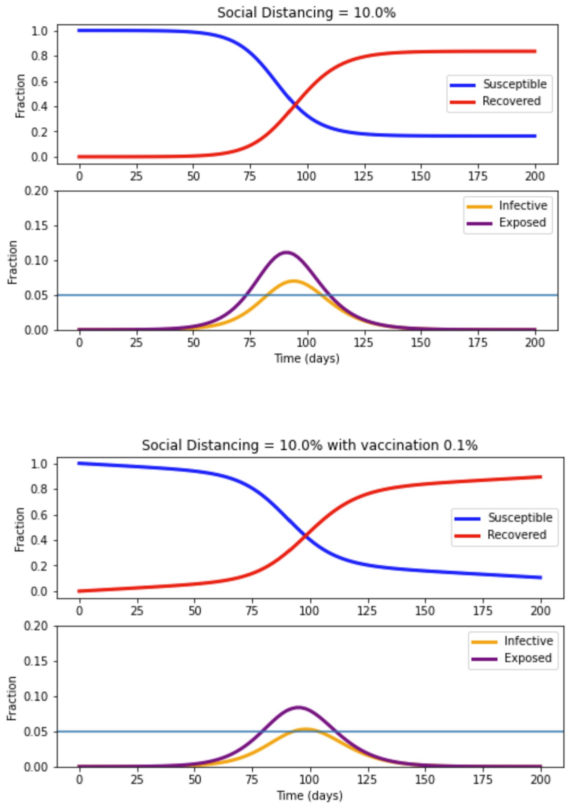

Figure 3: Social distancing parameter $\mu = 0.1$ and vaccination parameter $C = 0.1\% N$.

Source: Plotted by Python

From Figure 3, we find when $\mu = 0.1$ and 0.1% of the population can get vaccinated per day, the maximum of $\frac{I}{N}$ is about 0.05, which is smaller than the maximum when only having social distancing. Therefore, we can conclude that social distancing, together with COVID-19 vaccination programs, will help contain the epidemic more effectively.

3.3 Drawbacks of the ODE models and strategies for containing the epidemic

In this subsection, we are going to discuss several drawbacks of the previous ODE models and provide some strategies for minimizing disruption to the healthcare system base on the ODE models.

In terms of drawbacks of the previous ODE models, we would like to mention two important characteristics in epidemiology.

- Spatial heterogeneity

All of the ODE models we use don't take spatial heterogeneity into account. In other words, the previous models assume that the space and position would not influence the transmission of the disease. However, spatial heterogeneity actually plays an important role in epidemiology. For instance, the important parameter β (effective contact rate) is not uniform over the entire region. In urban areas, it is very likely to have a larger effective contact rate since it's more crowded. On the contrary, we may have a relatively small β in rural areas. Therefore, the neglect of spatial heterogeneity is one drawback of the previous ODE models.

- Temporal heterogeneity

Remember that β is the parameter used to characterize the rate at which individuals bump into others and expose them to the virus. For simplicity, we consider β as a constant in the above discussions. In reality, β is a function of time t rather than a constant. For example, we consider the different variants of the original COVID-19 virus. The Alpha, Beta, Delta, and Omicron variants have different values of R_0. In other words, they transmit in various speed. Correspondingly, we have different values of $\beta(t)$ as time goes by. In the case of COVID-19 epidemic, $\beta(t)$ tends to increase during some period as the coronavirus mutates. Therefore, the neglect of temporal heterogeneity is another drawback of the previous ODE models.

In addition, we would like to discuss the strategies for containing the epidemic effectively and at relatively low cost.

First of all, we feel the urge to make our principle of tackling COVID-19 epidemic very clear. The principle is to keep the number of infectious cases per day less than some threshold such as the capacity of hospitals rather than really achieve 'Zero Covid'.

Secondly, according to the previous results, we find that some certain level of social distancing like wearing masks in public places and vaccination programs are two effective ways to contain the epidemic.

Thirdly, too much social distancing like lockdown will cause many negative effects and is not necessary since with the help of vaccination we are able to keep the epidemic under control according to Figure 3.

Fourthly, note that for the majority (over 90 percent) of infected individuals Omicron appears to have mild symptoms or no symptoms at all. Therefore, we should keep calm instead of feeling terrified when facing the Omicron epidemic. At the same time, we still need to protect ourselves against the coronavirus such as getting the COVID-19 vaccine booster shot and keeping some certain level of social distancing.

Fifth, washing our hands often, covering our coughs, and avoiding gatherings are also effective ways to protect ourselves and others against coronavirus.

4 Conclusion

Compartmental models are a very general modelling technique. They are often applied to the mathematical modelling of infectious diseases. The population is assigned to compartments with labels and people may progress between compartments. In this report, we investigate the SIR model and some variants of it such as SEIR model and SEIR model with social distancing. The variant models are actually based on the SIR model. We can introduce new states and new parameters to the original SIR model in order to get a new variant.

The major results we would like to summarize are as follows:

(1) We investigate the properties and dynamics of SEIR model. The trends in the number of infected individuals per day can be characterized by exponential grow, reaching the maximum, and exponentail decay. When $\frac{S}{N} \approx 1$, the equilibrium $(0, 0)$ is a saddle; when $\frac{S}{N} \approx 0$, the equilibrium $(0, 0)$ is a stable node. One may refer to the paper by Ajbar et al. (2021) to see some results of the bifurcations of a generalized SIR model.

(2) We implement SEIR model with social distancing and vaccination using Python. We find that with the social distancing parameter $\mu = 0.1$ and 0.1% of the population getting vaccinated per

day we can keep the maximum of the number of infected individuals around 0.05% of the total population without further increasing the level of social distancing.

(3) We finally point out two drawbacks of the ODE models we use – the neglect of spatial heterogeneity and temporal heterogeneity. We also provide some strategies for containing the epidemic effectively based on the ODE models. For example, some certain level of social distancing like wearing masks in public places and vaccination programs are two effective ways to contain the epidemic.

In terms of the extensions and improvements, we would like to propose the following ideas:

(1) Further explore the dynamics of an SEIR-Based COVID-19 Model. For instance, we can investigate what the possible bifurcation is going to be and how the bifurcation depends on the different parameters.

(2) In terms of the SEIR model with social distancing, we can use some real data to estimate what the parameter μ represents exactly. For example, if $\mu = 0.1$, we can estimate what level of social distancing is going to be implemented in real life based on some data.

(3) One may introduce some new states or parameters like the birth and death rate to the SEIR model and investigate if the new model can fit the real-life scenario better.

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Appendix

A Code demo

Please see the complete code in 'ODE model Qixuan Wang.ipynb'.

```
# Simulate in a loop
for (i in 2:Rows) {
    dt$R[i] = max(0, dt$R[i-1]  +  gamma*dt$I[i-1]*dTime)

    dt$I[i] = max(0, dt$I[i-1]  +  (sigma*dt$E[i-1] - gamma*dt$I[i-1] )*dTime)

    dt$E[i] = max(0, dt$E[i-1]  +  ( (beta*dt$S[i-1]*dt$I[i-1]/N) - sigma*dt$E[i-1])*dTime)
    # Check S + E + I + R = 1
    if(dt$R[i] + dt$I[i] +  dt$E[i] > N) {
        dt$E[i] = max(0, N - dt$R[i] - dt$I[i])
    }

    dt$S[i] = max(0, dt$S[i-1]  -  (beta*dt$S[i-1]*dt$I[i-1]/N)*dTime)
    # Check S + E + I + R = 1
    if(dt$R[i] + dt$I[i] + dt$S[i] + dt$E[i] > N) {
    dt$S[i] = max(0, N - dt$R[i] - dt$I[i] - dt$E[i])
    }
}
```