MATH 303 Applied Project: Modeling the COVID-19 Epidemic with ODE

Qixuan Wang
Supervised by Prof. Konstantinos Efstathiou

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昆山杜克大学
DUKE KUNSHAN
UNIVERSITY

Outline

1. Introduction to ODE models for the epidemic
    - SIR model
    - Variants of the SIR model
2. Research questions
    - How do ODE models help us understand the epidemic?
    - How will social distancing and vaccination help contain the epidemic?
3. Results
    - Properties and dynamics of SEIR model
    - Implementation of SEIR model with social distancing and vaccination
    - Strategies for containing the epidemic based on ODE models
4. Summary

Big news

China Eases 'Zero Covid' Restrictions:

START THE DAY HERE

Democrats secure Senate win in Georgia. Trump Organization found guilty of criminal tax fraud. China scraps some Covid rules.

China scraps some of its most controversial Covid rules, in significant step toward reopening

REUTERS®

World Business Legal Markets More

China

5 minute read · December 7, 2022 7:51 PM GMT+8 · Last Updated an hour ago

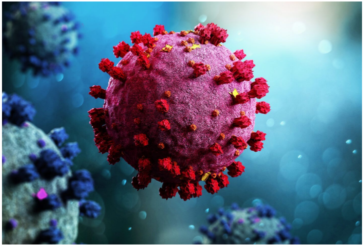
Chinese cheer as COVID curbs are loosened

By Martin Quin Pollard and Brenda Goh

Therefore, it's a perfect time to explore ODE models for the epidemic.

Introduction to ODE models for the epidemic

1. SIR model
2. Variants of the SIR model
   - SEIR model
   - SEIR model with social distancing
   - SEIR model with vaccination



https://www.woah.org/en/what-we-offer/emergency-preparedness/covid-19/

Introduction to ODE models for the epidemic

SIR model:

- The members of a population of size N fall into three classes:
  - $S(t)$ = the number of susceptible individuals—that is, those who have not been infected; $s := S/N$ is the fraction of susceptibles.
  - $I(t)$ = the number of individuals who are currently infected, comprising a fraction $i := I/N$ of the population.
  - $R(t)$ = the number of individuals who have recovered from infection, comprising the fraction $r := R/N$.

https://dl.acm.org/doi/fullHtml/10.1145/3449258.3449261

Introduction to ODE models for the epidemic

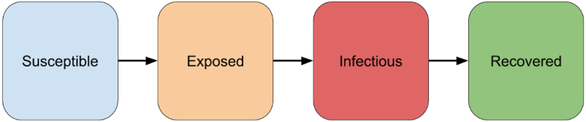
SIR model can be expressed by the following system of ODEs:

$$\begin{cases} \dfrac{dS}{dt} = -\dfrac{\beta IS}{N}, \\ \dfrac{dI}{dt} = \dfrac{\beta IS}{N} - \gamma I, \\ \dfrac{dR}{dt} = \gamma I, \end{cases}$$

- Note that $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$.
- Those who have recovered, and live, are assumed to have acquired immunity.
- An ideal model.

Introduction to ODE models for the epidemic

SEIR model:



$$\frac{dS}{dt} = -\beta S \frac{I}{N}$$
$$\frac{dE}{dt} = \beta S \frac{I}{N} - \sigma E$$
$$\frac{dI}{dt} = \sigma E - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

Introduction to ODE models for the epidemic

SEIR model:

$$
\begin{aligned}
\frac{dS}{dt} &= -\beta S \frac{I}{N} \\
\frac{dE}{dt} &= \beta S \frac{I}{N} - \sigma E \\
\frac{dI}{dt} &= \sigma E - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{aligned}
$$

- Note that $\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$.
- In practice, the infection rate $\sigma \approx \frac{1}{\text{incubation period}}$.
- The recovery rate $\gamma \approx \frac{1}{\text{duration infection}}$.
- The basic reproduction number $R_0 \approx \frac{\beta}{\gamma}$ where β is the effective contact rate.

Introduction to ODE models for the epidemic

SEIR model with social distancing:

$$\frac{dS}{dt} = -(1-u)\frac{\beta SI}{N}$$
$$\frac{dE}{dt} = (1-u)\frac{\beta SI}{N} - \alpha E$$
$$\frac{dI}{dt} = \alpha E - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

- Note that $\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$.
- μ describes the effectiveness on any public health interventions to control transmission of the disease.
- $\mu \in [0, 1]$.
- $0 =$ no social distancing; $1 =$ complete isolation.

Introduction to ODE models for the epidemic

SEIR model with vaccination:

$$\frac{dS}{dt} = -\beta S \frac{I}{N} - C$$
$$\frac{dE}{dt} = \beta S \frac{I}{N} - \sigma E$$
$$\frac{dI}{dt} = \sigma E - \gamma I$$
$$\frac{dR}{dt} = \gamma I + C$$

- Note that $\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$.
- C is the number of people who get vaccinated in one day.
- Once people get vaccinated, they are assumed to have acquired immunity.
- One drawback: the above models ignore the differences between the regions.

Research questions

Research questions:

- How do ODE models help us understand the epidemic?
- How will social distancing and vaccination help contain the epidemic?

https://ohiostate.pressbooks.pub/choosingsources/chapter/purpose-of-research-questions/

Results (Part I)

Properties of SEIR model:

$$\frac{dS}{dt} = -\beta\, S\, \frac{I}{N}$$
$$\frac{dE}{dt} = \beta\, S\, \frac{I}{N} - \sigma E$$
$$\frac{dI}{dt} = \sigma E - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

Case 1: $\frac{S}{N} \approx 1$ (start)

- $\frac{dE}{dt} = \beta I - \sigma E$, $\frac{dI}{dt} = -\gamma I + \sigma E$.

Results (Part I)

Case 1: $\frac{S}{N} \approx 1$ (start)

- $\frac{dE}{dt} = \beta I - \sigma E$, $\frac{dI}{dt} = -\gamma I + \sigma E$.
- A linear system with constant coefficients.
- The only equilibrium is $(0, 0)$ (a saddle).

$$\frac{d}{dt}\begin{bmatrix} E \\ I \end{bmatrix} = \begin{bmatrix} -\sigma & \beta \\ \sigma & -\gamma \end{bmatrix}\begin{bmatrix} E \\ I \end{bmatrix}$$

The characteristic polynomial: $(-\sigma - \lambda)(-\gamma - \lambda) - \sigma\beta = 0 \Rightarrow$

$\lambda^2 + (\sigma + \gamma)\lambda - \sigma(\beta - \gamma) = 0 \Rightarrow$

$\Delta = (\sigma + \gamma)^2 + 4\sigma(\beta - \gamma) = (\sigma - \gamma)^2 + 4\sigma\beta > 0 \Rightarrow$

We have two real eigenvalues: $\lambda_+\lambda_- = -\sigma(\beta - \gamma) < 0$

$\beta = R_0\gamma$, $R_0 > 1 \Rightarrow \beta > \gamma$

Hence we will have exponential growth in E and I due to the positive λ_+.

Results (Part I)

Case 2: $\frac{S}{N} \approx 0$ (end)

- $\frac{dE}{dt} = -\sigma E$, $\frac{dI}{dt} = -\gamma I + \sigma E$.
- A linear system with constant coefficients.
- The only equilibrium is $(0, 0)$ (a stable node).

$$\frac{d}{dt}\begin{bmatrix} E \\ I \end{bmatrix} = \begin{bmatrix} -\sigma & 0 \\ \sigma & -\gamma \end{bmatrix}\begin{bmatrix} E \\ I \end{bmatrix}$$

Therefore, $\lambda_1 = -\sigma$, $\lambda_2 = -\gamma$

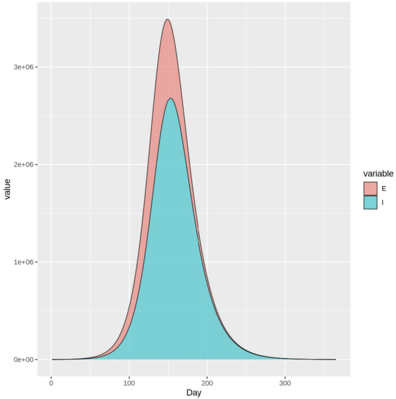
Hence we will have exponential decay in E and I due to negative eigenvalues.

Results (Part I)

Case 3: $\frac{dE}{dt} \approx 0$, $\frac{dI}{dt} \approx 0$ (stop growing; herd immunity is reached)

- $\gamma I \approx \sigma E$, $\beta S \frac{I}{N} \approx \sigma E$
- Hence $\gamma I \approx \beta S \frac{I}{N} \Rightarrow$
- $\frac{S}{N} \approx \frac{\gamma}{\beta}$

Therefore, I is maximal when $\frac{S}{N} \approx \frac{\gamma}{\beta}$.



Results (Part II)

Implementation of SEIR model with social distancing:

$$\frac{dS}{dt} = -(1-u)\frac{\beta SI}{N}$$
$$\frac{dE}{dt} = (1-u)\frac{\beta SI}{N} - \alpha E$$
$$\frac{dI}{dt} = \alpha E - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

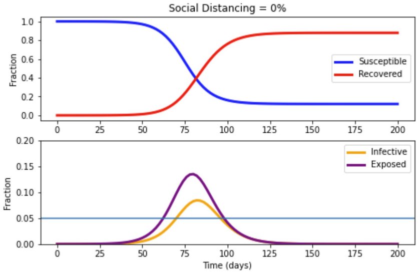
μ: social distancing $\in [0, 1]$

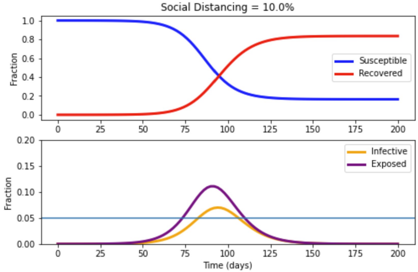
- $0 =$ no social distancing
- $0.1 =$ masks
- $0.2 =$ masks and hybrid classes
- $0.3 =$ masks and online classes

We don't want μ to go above 0.3*. Otherwise, it would be too much for us.*

Results (Part II)

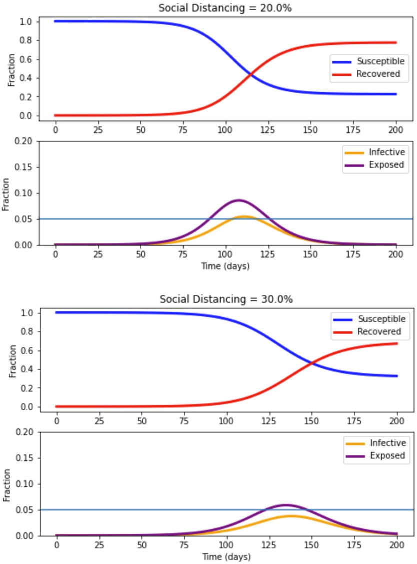
Implementation of SEIR model with social distancing:





Results (Part II)

Implementation of SEIR model with social distancing:



Results (Part II)

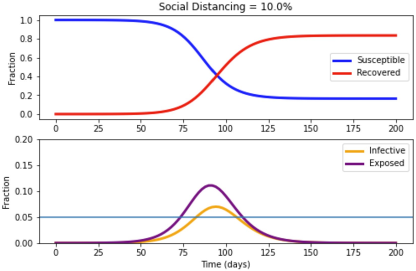
Implementation of SEIR model with social distancing and vaccination (a relatively new model):

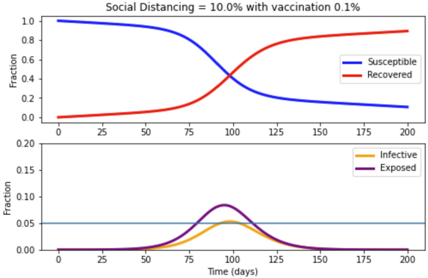
- Assume 0.1% of the population would get vaccinated everyday, which means $C = 0.1\% N$.

$$\frac{dS}{dt} = -(1-u)\frac{\beta SI}{N} - C$$
$$\frac{dE}{dt} = (1-u)\frac{\beta SI}{N} - \alpha E$$
$$\frac{dI}{dt} = \alpha E - \gamma I$$
$$\frac{dR}{dt} = \gamma I + C$$

Results (Part II)

Implementation of SEIR model with social distancing and vaccination:





Result (Part III)

Discussion and strategies for tackling the epidemic::

1. The principle is not to achieve 'Zero Covid', but to keep the infectious cases less than some threshold (the capacity of hospitals).
2. Some certain level of social distancing is necessary, but too much social distancing like ~~lockdown~~ will cause many negative effects and is not necessary.
3. Wearing masks in public places and vaccination are indeed two effective ways to contain the epidemic.
4. Omicron is not terrifying. For over 90% of the patients, Omicron appears to have mild symptoms or no symptoms at all.

Summary

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3. Results
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   - Wearing masks in public places and vaccination are two effective ways to contain the epidemic

A few words

A few words:

- Several months ago, the end of COVID epidemic in China was nowhere in sight.
- Now we finally see the light at the end of the tunnel.
- However, we should not forget the great efforts and sacrifices made by the medical staff and scientists.
- Keep calm and protect ourselves. The end of COVID epidemic is in sight!

Reference

References:

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- Fundamentals of Differential Equations
- Modeling COVID 19 with Differential Equations: https://julia.quantecon.org/continuous-time/seir-model.htmlid6
- SEIR - models: properties: https://www.youtube.com/watch?v=JGhfGHuJuJc

Thanks for your listening!