An elementary proof of the law of iterated logarithm for minima and new extensions of the Borel-Cantelli Lemma

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Abstract

In this note we present a new, simple proof of the law of iterated logarithm for minima of uniform random variables. We also prove new extensions of the Borel-Cantelli Lemma.
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1. Introduction

Let $X, X_1, X_2, \cdots$ be independent identically distributed random variables, and let $W_n = \min\{X_1, \cdots, X_n\}$. Following the terminology attributed to Paul Lévy, we say that a sequence of positive numbers, say $\{v_n\}$, is said to be a lower class sequence for W_n if the event $\{W_n \geqslant v_n\}$ occurs infinitely often with probability one. A contribution of these notes is a new, simple proof of the following series characterization criterion for such sequences.

Note: Throughout the rest of the paper we write $log_2 = \log\log$, and, unless otherwise stated, all limits are to be interpreted with respect to n, as $n \to +\infty$.

Theorem 1. *Let $X_1, X_2, \cdots, X_n$ be independent and uniform random variables on $(0,1)$, and let $W_n = \min\{X_1, \cdots, X_n\}$. Let $v_n = \frac{c_n}{n}$ be a sequence of positive real numbers, with v_n non-increasing for all sufficiently large n.*

If $\liminf \dfrac{c_n}{\log_2 n} \geqslant 1$, *then*

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$$P\left(W_n \geqslant v_n \; i.o.\right) = \begin{cases} 0 & if \quad \sum_{n=1}^{+\infty} P\left(W_n \geqslant v_n,\, W_{n+1} < v_n\right) < +\infty \\ 1 & if \quad \sum_{n=1}^{+\infty} P\left(W_n \geqslant v_n,\, W_{n+1} < v_n\right) = +\infty. \end{cases}$$

Theorem 1 is due to Robbins and Siegmund (1970). A similar result had been previously obtained by Barndorff-Nielson (1961) under the more restrictive requirement that c_n is eventually non-decreasing. Both papers employed the same method of proof used by Erdös (1942) in his proof of the general form of the law of the iterated logarithm. As far as we know there is no other simpler proof of this theorem. An attempt in this direction can be found in Stepanov (2014), but we believe that the proof is based on a false assumption.

A sequence of events $\{A_n\}$ is said to be a Markov sequence if the sequence of Bernoulli random variables $\{I_{A_n}\}$ is a Markov chain. Stepanov (2014) proves the following result.

Lemma 1. *Let $\{A_n\}$ be a Markov sequence such that $P(A_n) \to 0$. Then*

$$P\left(A_n \, i.o.\right) = \begin{cases} 0 & if \quad \sum_{n=1}^{+\infty} P\left(A_n \cap A_{n+1}^c\right) < +\infty \\ 1 & if \quad \sum_{n=1}^{+\infty} P\left(A_n \cap A_{n+1}^c\right) = +\infty. \end{cases}$$

Then they use this result to justify the validity of Theorem 1. Their proof relies on the assumption that the events $A_n = \{W_n \geqslant v_n\}$ form a Markov sequence, which guarantees that for every $n, k \geqslant 1$,

$$P(A_{n+k}^c | A_{n+k-1}^c \cap \cdots \cap A_n^c) = P(A_{n+k}^c | A_{n+k-1}^c). \tag{1}$$

However $I_{\{W_n \geqslant v_n\}}$ is a function of W_n, and hence it is not necessarily a Markov chain even if W_n is; we provide two examples.

Example 1. *Let X be a discrete uniform random variable on $S = \{1, 2, 3, 4\}$, $\{a_n\}$ a non-increasing sequence with values in S, and $A_n = \{W_n \geqslant a_n\}$. Let $a_1 = 4, a_2 = 3, a_3 = 2$, for example, then*

$$\begin{aligned} P(A_3^c | A_2^c \cap A_1^c) &= \frac{P(A_3^c \cap A_2^c \cap A_1^c)}{P(A_2^c \cap A_1^c)} \\ &= \frac{1 - P(A_1 \cup A_2 \cup A_3)}{1 - P(A_2 \cup A_1)} \\ &= \frac{1 - \left(\frac{1}{4} + \frac{4}{16} + \frac{27}{64} - \frac{2}{16} - \frac{9}{64} - \frac{12}{64} + \frac{6}{64}\right)}{1 - \left(\frac{1}{4} + \frac{4}{16} - \frac{2}{16}\right)} = \frac{7}{10}, \end{aligned}$$

whereas

$$\begin{aligned} P(A_3^c|A_2^c) &= \frac{P(A_3^c \cap A_2^c)}{P(A_2^c)} = \frac{1-P(A_2 \cup A_3)}{1-P(A_2)} \\ &= \frac{1-\left(\frac{4}{16}+\frac{27}{64}-\frac{12}{64}\right)}{1-\frac{4}{16}} = \frac{11}{16}. \end{aligned}$$

Hence $\{A_n\}$ *is not a Markov sequence.*

Example 2. *Let* X *be a uniform random variable on* $(0,1)$, $\{b_n\}$ *a decreasing sequence of real numbers with values on* $(0,1)$, *and let* $B_n = \{W_n \geqslant b_n\}$. *For* $n = 1, 2, 3$,

$$P(B_3^c|B_2^c \cap B_1^c) = P(B_3^c|B_2^c)$$

$$\iff P(B_2^c)\, P(B_3^c \cap B_2^c \cap B_1^c) - P(B_1^c \cap B_2^c)\, P(B_2^c \cap B_3^c) = 0.$$

We compute the two terms on the left hand side of the second equation, one at a time.

$$(i) \qquad P(B_2^c)\, P(B_3^c \cap B_2^c \cap B_1^c) = [1-P(B_2)]\,[1-P(A_1 \cup A_2 \cup A_3)]$$

$$= \Big[1-(1-b_2)^2\Big]\Big[1-\Big((1-b_1)+(1-b_2)^2+(1-b_3)^3-(1-b_1)(1-b_2)$$

$$-(1-b_1)(1-b_3)^2-(1-b_2)^2(1-b_3)+(1-b_1)(1-b_2)(1-b_3)\Big)\Big],$$

$$(ii) \qquad P(B_1^c \cap B_2^c)\, P(B_2^c \cap B_3^c) = \Big(1-P(B_1 \cup B_2)\Big)\Big(1-P(B_2 \cup B_3)\Big)$$

$$= \Big[1-\Big((1-b_1)+(1-b_2)^2-(1-b_1)(1-b_2)\Big)\Big]\Big[1-\Big((1-b_2)^2$$

$$+(1-b_3)^3-(1-b_2)^2(1-b_3)\Big)\Big].$$

For convenience in (i) and (ii) we let $b_1 = 3x$, $b_2 = 2x$, *and* $b_3 = x$, *and with these values after simplification we obtain*

$$\begin{aligned} &P(B_2^c)\, P(B_3^c \cap B_2^c \cap B_1^c) - P(B_1^c \cap B_2^c)\, P(B_2^c \cap B_3^c) \\ =\ &2x^3 - 8x^4 + 6x^5 \not\equiv 0 \quad in\ (0, 1/3). \end{aligned}$$

Hence $\{B_n\}$ is not a Markov sequence.

Remark 1. *The proof of Lemma 1 does not rely on the full assumption that $\{A_n\}$ is a Markov sequence, but only on the validity of (1). Example 2 shows that the events $\{W_n \geqslant v_n\}$ do not satisfy (1), and hence Theorem 1 does not follow from Stepanov's version of the Borel-Cantelli lemma applicable to Markov sequences.*

In Section 2 we present our new proof of Theorem 1, and show how it can be applied indirectly to sequences with lim sup bounded away from 1. In Section 3 we propose new extensions of the Borel-Cantelli Lemma.

2. Alternative proof of Theorem 1

Before presenting our proof we make a few observations.

Remark 2. *It follows from our calculations that the conclusion of Theorem 1 continues to hold if the condition* $\liminf \dfrac{c_n}{\log_2 n} \geqslant 1$ *is replaced with the more general condition* $\liminf \dfrac{c_n}{\log_2 n} \geqslant \delta > 0$.

Remark 3. *The assumption that the X_i's are uniform random variable on $(0,1)$ is not a limitation. In fact Theorem 1 implies a law of iterated logarithm for arbitrary i.i.d. random variables with a continuous distribution, see Remark 2.5 in Robbins and Siegmund (1970), and Chapter 4 of Galambos (1970) for a thorough discussion of the general case.*

It is important to mention that Theorem 1 can be applied indirectly to any sequence c_n/n such that $\limsup \dfrac{c_n}{\log_2 n} = \alpha < 1$, as a consequence of the following result.

Lemma 2. *Let $X_1, X_2, \cdots, X_n$ be independent and uniform random variables on $(0,1)$, and let $W_n = \min\{X_1, \cdots, X_n\}$. Let $u_n = c_n/n$ be a sequence of positive real numbers.*

If $\limsup \dfrac{c_n}{\log_2 n} = \alpha < 1$, *then*

$$P\left(W_n \geqslant u_n \ i.o.\right) = 1.$$

Next we turn to the proof of Theorem 1. Our proof is elementary and for the most part self contained. The only additional result we need is the following inequality due to Feng and alt.

(2009), which is a weighted version of a result of Erdös and Rényi, see Lemma C in Erdös and Rényi (1959).

Lemma 3. *Let B_n be any sequence of events in a probability space $(\Omega, \mathcal{A}, P)$, and $\{w_n\}$ a sequence of real numbers. If*

$$\sum_{k=1}^{+\infty} w_k P(B_k) = +\infty,$$

then for any positive integer s

$$P\left(B_n\, i.o.\right) \geqslant \limsup \frac{\left(\sum_{k=s}^{n} w_k P(B_k)\right)^2}{\sum_{k=s}^{n} \sum_{m=s}^{n} w_k\, w_m\, P\left(B_k \cap B_m\right)}.$$

Proof of Theorem 1. Let $A_n = \{W_n \geqslant v_n,\, W_{n+1} < v_n\}$, $B_n = \{W_n \geqslant v_n\}$ with $P(B_n) \to 0$. We first assume that

$$\sum_{n=1}^{+\infty} P(A_n) = \sum_{n=1}^{+\infty} v_n(1-v_n)^n < \infty.$$

Our next steps are similar to the ones used in N. Balakrishnan and A. Stepanov (2010), see also T.K. Chandra (2012).

$$\begin{aligned} P\Big(B_n\ i.o.\Big) &\leqslant P\left(\bigcup_{k=n}^{+\infty} B_k\right) = \lim_{t\to+\infty} P\left(\bigcup_{k=n}^{n+t} B_k\right) \leqslant \lim_{t\to+\infty}\left(P(B_{n+t}) + \sum_{j=0}^{t-1} P\Big(B_{n+j}\cap B_{n+j+1}^c\Big)\right) \\ &= \sum_{k=n}^{+\infty} P\Big(B_k \cap B_{k+1}^c\Big) = \sum_{k=n}^{+\infty} P\Big(A_k\Big). \end{aligned}$$

Since the above inequality holds for every n, by letting $n \to +\infty$ we get $P\Big(B_n\ i.o.\Big) = 0$.

Next we assume that

$$\sum_{n=1}^{+\infty} P(A_n) = \sum_{n=1}^{+\infty} v_n(1-v_n)^n = +\infty. \tag{2}$$

Under this assumption we can further assume that $v_n \leqslant 2\log_2(n)/n$. In fact, suppose there exists a subsequence of $\{v_n\}$, say $\{v_{n_k}\}$, such that for every $k \geqslant 1$, $v_{n_k} > 2\log_2(n_k)/n_k$. Since

$$\sum_{k=1}^{+\infty} P\left(W_{n_k} \geqslant 2\,\frac{\log_2(n_k)}{n_k},\, W_{n_{k+1}} < 2\,\frac{\log_2(n_k)}{n_k}\right)$$
$$\leqslant \sum_{n=3}^{+\infty} 2\,\frac{\log_2(n)}{n}\left(1 - 2\,\frac{\log_2(n)}{n}\right)^n \leqslant \sum_{n=3}^{+\infty} 2\,\frac{\log_2(n)}{n}\,\frac{1}{\log(n)^2} < +\infty,$$

then from the first part of our proof we have $P\left(W_{n_k} \geqslant 2\,\frac{\log_2(n_k)}{n_k}\, i.o.\right) = 0$. This combined with

$$P\left(W_{n_k} \geqslant v_{n_k}\, i.o.\right) \leqslant P\left(W_{n_k} \geqslant 2\,\frac{\log_2(n_k)}{n_k}\, i.o.\right)$$

imply $P(W_{n_k} \geqslant v_{n_k}\, i.o.) = 0$. Hence

$$P\Big(W_n \geqslant v_n\ i.o\Big) = P\Big(W_n \geqslant \min\{v_n, 2\log_2(n)/n\}\Big).$$

Since for this new sequence (2) continues to hold, we may assume, without loss of generality, that

$$\frac{1}{2}\,\log_2(n) \leqslant c_n \leqslant 2\,\log_2(n). \tag{3}$$

By Lemma 3, with B_n as in the fist part of our proof and $w_n = v_n$, we have that for every positive integer s

$$P\Big(B_n\, i.o.\Big) \geqslant \limsup \frac{\left(\sum_{k=s}^{n} v_k(1-v_k)^k\right)^2}{\sum_{i=s}^{n} v_i^2(1-v_i)^i + 2\sum_{i=s}^{n}\sum_{j=i+1}^{n} v_i(1-v_i)^i\, v_j(1-v_j)^{j-i}}. \tag{4}$$

Let

$$S(n) = \sum_{k=s}^{n} v_k(1-v_k)^k.$$

We claim that the lower bound in (4) equals 1. Let $\beta \geqslant 3$ be arbitrary but fixed. We decompose the double sum in the denominator of (4) into three parts,

$$\sum_{i=s}^{n}\sum_{j=i}^{n}(\cdots) = \underbrace{\sum_{i=s}^{n}\sum_{j=i+1}^{t(i)}(\cdots)}_{S_1(n)} + \underbrace{\sum_{i=s}^{n}\sum_{j=t(i)+1}^{r(i)}(\cdots)}_{S_2(n)} + \underbrace{\sum_{i=s}^{n}\sum_{j=r(i)+1}^{n}(\cdots)}_{S_3(n)}, \tag{5}$$

where $t(i) = 4i$, and $r(i) = 4\beta i \log_2(i)$. Let s be the smallest positive integer such that if $i \geqslant s$, the following properties hold:

$$\begin{array}{ll} (a) & \frac{\log_2(i)}{i}\downarrow \\ (b) & i\,2\,\frac{\log_2(r(i))}{r(i)} < \frac{1}{\beta}. \end{array} \tag{6}$$

Note that s is well defined. We analyze each of the the sums in (5) separately, starting with $S_1(n)$.

$$\begin{aligned} S_1(n) &= \sum_{i=s}^{n} v_i(1-v_i)^i \sum_{j=i+1}^{4i} v_j(1-v_j)^{j-i} \\ &\leqslant \sum_{i=s}^{n} v_i(1-v_i)^i \sum_{j=i+1}^{4i} 2\,\frac{\log_2(j)}{j}\left(1-\frac{\log_2(j)}{2j}\right)^{j-i} \\ &\leqslant \sum_{i=s}^{n} v_i(1-v_i)^i \sum_{j=0}^{+\infty} 2\,\frac{\log_2(i)}{i}\left(1-\frac{\log_2(4i)}{8i}\right)^{j} \\ &\leqslant 16\sum_{i=s}^{n} v_i(1-v_i)^i = 16\,S(n). \end{aligned}$$

We analyze $S_2(n)$ next.

$$\begin{aligned} S_2(n) &= \sum_{i=s}^{n} v_i(1-v_i)^i \sum_{j=4i+1}^{r(i)} v_j(1-v_j)^{j-i} \\ &\leqslant \sum_{i=s}^{n} v_i(1-v_i)^i \sum_{j=4i+1}^{r(i)} 2\,\frac{\log_2(4i)}{4i}\,(e^{-c_j})^{(j-i)/j} \\ &\leqslant \sum_{i=s}^{n} v_i(1-v_i)^i \left[r(i)\,2\,\frac{\log_2(4i)}{4i}\,\frac{1}{\log(4i)^{1/4}}\right] \qquad (7) \\ &\leqslant \sum_{i=s}^{n} v_i(1-v_i)^i \left[2\beta\,\log_2(4i)^2\,\frac{1}{\log(4i)^{1/4}}\right] \qquad (8) \end{aligned}$$

where in (7) we used the inequality $(j-i)/j > 1/2$ which holds for $j > 4i$. Since the quantity inside the square bracket in (8) goes to 0 as i goes to $+\infty$, there exists a constant c such that

$$S_2(n) \leqslant c\, S(n).$$

It remains to analyze $S_3(n)$,

$$S_3(n) = \sum_{i=s}^{n} v_i(1-v_i)^i \sum_{j=r(i)+1}^{n} v_j(1-v_j)^j(1-v_j)^{-i}.$$

Using

$$iv_j \leqslant i\,2\,\frac{\log_2(j)}{j} \leqslant i\,2\,\frac{\log_2(r(i))}{r(i)} < \frac{1}{\beta},$$

we obtain

$$(1-v_j)^{-i} = \sum_{k=0}^{+\infty} \binom{i+k-1}{k} v_j^k \leqslant \sum_{k=0}^{+\infty} \frac{1}{\beta^k} = \frac{\beta}{\beta-1},$$

and therefore

$$\begin{aligned} S_3(n) &\leqslant \sum_{i=s}^{n} v_i(1-v_i)^i \sum_{j=r(i)+1}^{n} v_j(1-v_j)^j \frac{\beta}{\beta-1} \\ &\leqslant \frac{\beta}{\beta-1}\frac{S(n)^2}{2}. \end{aligned}$$

Using the bounds we derived for $S_i(n)$, $i = 1, 2, 3$ we obtain

$$P\Big(W_n \geqslant v_n\, i.o.\Big) \geqslant \limsup \ \frac{S(n)^2}{(1+32+2c)\,S(n) + \frac{\beta}{\beta-1}S(n)^2} = \frac{\beta-1}{\beta}.$$

Since β was arbitrary, and $\beta/(\beta-1) \to 1$ as β goes to $+\infty$, the above calculations imply

$$P\Big(W_n \geqslant v_n\, i.o.\Big) = 1.$$

The proof of the theorem is now complete.

□

Remark 4. *It follows from the first part of the proof of Theorem 1 that if a sequence of events, say* $\{E_n\}$, *is such that* $P(E_n) \to 0$ *and*

$$\sum_{n=1}^{+\infty} P\Big(E_n \cap E_{n+1}^c\Big) < +\infty,$$

then $P(E_n\, i.o.) = 0$. *This result is due to Barndorff-Nielson (1961).*

Proof of Lemma 2. Let $u_n = c_n/n$, and suppose

$$\limsup \frac{c_n}{\log_2 n} = \alpha < 1.$$

Let $a_n = \frac{\log_2(n)}{n}$, and $C_n = \{W_n \geqslant a_n\}$.

$$\sum_{n=1}^{+\infty} P(C_n \cap C_{n+1}^c) \leqslant \sum_{n=1}^{+\infty} \frac{\log_2(n)}{n} \frac{1}{\log(n)} = +\infty,$$

and by Theorem 1, $P\Big(C_n\, i.o.\Big) = 1$. For all n sufficiently large $C_n \subset \{W_n \geqslant u_n\}$, and thus,

$$P\Big(W_n \geqslant u_n\, i.o.\Big) \geqslant P\Big(C_n\, i.o.\Big) = 1.$$

□

3. A generalization of Lemma 1

In Lemma 1, the assumption that $\{A_n\}$ is a Markov sequence is only needed if $\sum_{n=1}^{+\infty} P(A_n \cap A_{n+1}^c) = +\infty$, see Remark 4. When this holds the assumption $P(A_n) \to 0$ is not needed. Furthermore, as we pointed out in Remark 1, one can relax the requirement that $\{A_n\}$ is a Markov sequence and simply require that (1) holds. We propose the following generalization of Lemma 1.

We confine ourselves to the non-trivial case $\limsup P(A_n) < 1$, because if $\limsup P(A_n) = 1$, then $P(A_n\, i.o.) = 1$ immediately follows, see beginning of proof of Lemma 4 for details, regardless of whether $\sum P(A_n \cap A_{n+1}^c)$ converges or diverges.

Lemma 4. *Let $\{A_n\}$ be a sequence of events, and suppose that* $\limsup P(A_n) = a < 1$. *If*

$$\sum_{n=1}^{+\infty} P\left(A_n \cap A_{n+1}^c\right) < +\infty, \tag{9}$$

then

$$P(A_n \, i.o.) = a. \tag{10}$$

Moreover if there exists a positive integer N such that for $n \geqslant N$ and $k \geqslant 1$,

$$P(A_{n+k}^c \,|\, A_{n+k-1}^c \cap \cdots \cap A_n^c) \leqslant P(A_{n+k}^c \,|\, A_{n+k-1}^c), \tag{11}$$

then

$$P\left(A_n \, i.o.\right) = 1 \quad \text{ if and only if } \quad \sum_{n=1}^{+\infty} P\left(A_n \cap A_{n+1}^c\right) = +\infty. \tag{12}$$

Note that the first part of Lemma 4 generalizes the result of Barndorff-Nielson (1961) mentioned in Lemma 4. If $\{A_n\}$ is an increasing sequence of events then (11) trivially holds, and the second part generalizes a result of F.T. Bruss (1980).

Proof. Let $\{A_n\}$ be a sequence of events and suppose $\limsup P(A_n) = a < 1$. Let $\{t_n\} \subset \{n\}$, $t_n > n$, and $P(A_{t_n}) \to a$. Then

$$P(A_n \, i.o.) = \lim P\left(\bigcup_{k=n}^{+\infty} A_k\right) \geqslant \lim P(A_{t_n}) = a. \tag{13}$$

Next we suppose that (9) holds. By proceeding as in the first part of the proof of Theorem 1, we skip few details here, we obtain

$$\begin{aligned} P(A_n \, i.o.) &\leqslant \limsup_{t \to +\infty} \left(P(A_{n+t}) + \sum_{j=0}^{t-1} P\Big(A_{n+j} \cap A_{n+j+1}^c\Big)\right) \\ P(A_n \, i.o.) &\leqslant a + \sum_{k=n}^{+\infty} P\Big(A_k \cap A_{k+1}^c\Big). \end{aligned} \tag{14}$$

Since (14) holds for every n, by letting $n \to +\infty$ we immediately have

$$P(A_n\, i.o.) \leqslant a. \tag{15}$$

Thus $P(A_n\, i.o.) = a$.

Next we turn to the proof of (12). Since (15) proves the sufficiency part, to complete the proof it suffices to consider the case that (11) holds and that

$$\sum_{n=1}^{+\infty} P\left(A_n \cap A_{n+1}^c\right) = +\infty.$$

Since for arbitrary $t > 1$,

$$\begin{aligned}
\sum_{n=1}^{t} P\left(A_n \cap A_{n+1}^c\right) &= \sum_{n=1}^{t}\Big[1 - P(A_n^c \cup A_{n+1})\Big] \\
&= \sum_{n=1}^{t}\Big[1 - \Big(1 - P(A_n) + P(A_{n+1}) - P(A_n^c \cap A_{n+1})\Big)\Big] \\
&= P(A_1) + \sum_{n=1}^{t} P(A_n^c \cap A_{n+1}) - P(A_{t+1}),
\end{aligned}$$

we also have that

$$\sum_{n=1}^{+\infty} P\left(A_n^c \cap A_{n+1}\right) = +\infty.$$

Let $n \geqslant N$. Then

$$1 - P(A_n \cup A_{n+1} \cup \cdots \cup A_{n+k}) = P(A_n^c \cap A_{n+1}^c \cap \cdots \cap A_{n+k}^c) \tag{16}$$

$$= P(A_n^c)\, P(A_{n+1}^c|A_n^c) P(A_{n+2}^c|A_{n+1}^c \cap A_n^c) \cdots P(A_{n+k}^c|A_{n+k-1}^c \cap \cdots \cap A_n^c)$$

$$\leqslant P(A_n^c)\, P(A_{n+1}^c|A_n^c) P(A_{n+2}^c|A_{n+1}^c) \cdots P(A_{n+k}^c|A_{n+k-1}^c) \tag{17}$$

$$= e^{\log P(A_n^c) + \sum_{j=1}^{k} \log P(A_{n+j}^c|A_{n+j-1}^c)} = e^{\log P(A_n^c) + \sum_{j=1}^{k} \log(1 - P(A_{n+j}|A_{n+j-1}^c))}$$

$$\leqslant e^{\log P(A_n^c) - \sum_{j=1}^{k} P(A_{n+j}|A_{n+j-1}^c)} \leqslant e^{\log P(A_n^c) - \sum_{j=1}^{k} P(A_{n+j} \cap A_{n+j-1}^c)}, \tag{18}$$

where (17) follows from (11), and (18) follows from the elementary inequalities $\log(1-x) \leqslant -x$,

$0 < x < 1$, and $-P(A_{n+j}|A^c_{n+j-1}) \leqslant -P(A_{n+j} \cap A^c_{n+j-1})$. Next we let $k \to +\infty$ in (16) and (18), and obtain that for every $n \geqslant N$,

$$1 - P\Big(\bigcup_{i=n}^{+\infty} A_i\Big) \leqslant 0,$$

which further implies that

$$P(A_n\, i.o.) = \lim P\Big(\bigcup_{i=n}^{+\infty} A_i\Big) = 1.$$

The proof of the lemma is now complete.

□

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